Table 1 Comparison of calculated and experimental values of C_f and R_δ

		C_f		$R_\delta \times 10^{-4}$	
$\bar{V}_w$	$R_\theta \times 10^{-3}$	Experimental (Ref. 7)	Calculated	Experimental (Ref. 7)	Calculated
0	1.971	0.00392	0.00397	1.88	1.89
	4.318	0.00326	0.00317	3.88	3.97
−0.00118	1.238	0.00520	0.00539	1.28	1.31
	2.94	0.00440	0.00437	3.06	2.90
−0.00242	1.015	0.00622	0.00671	1.285	1.20
	1.738	0.00566	0.00596	2.27	1.90
−0.00465	0.191	0.00920	0.00922	0.39	0.22
0.00099	2.071	0.00336	0.00332	1.71	1.91
	4.887	0.00272	0.00255	4.02	4.30
0.00195	3.318	0.00244	0.00242	2.56	2.84
	6.583	0.00202	0.00201	5.05	5.58
0.00388	4.518	0.00148	0.00163	3.10	3.59
	9.429	0.00114	0.00147	6.62	7.37
0.00780	9.732	0.00036	0.00094	5.87	6.97
	15.935	0.00024	0.00082	10.00	11.50

References

[1] Van Driest, E. R., "On Turbulence Flow Near a Wall," *Journal of the Aeronautical Sciences,* Vol. 23, Nov. 1956, pp. 1007–1011.

[2] Cebeci, T., "The Behavior of Turbulent Flow Near a Porous Wall with Pressure Gradient," DAC-70014, May 1970, Douglas Aircraft Co., Long Beach, Calif.

[3] Koop, C. G., "Calculation of the Velocity and Shear-stress Profiles for Flat-Plate Incompressible Turbulent Boundary-Layer Flows With and Without Mass Transfer," M.S. thesis, 1971, Mechanical Engineering Dept., California State University, Long Beach, Calif.

[4] Coles, D., "The Law of the Wake in the Turbulent Boundary-Layer," *Journal of Fluid Mechanics,* Vol. 1, Pt. 1, June 1956, pp. 191–226.

[5] Wooldridge, C. E. and Muzzy, R. J., "Boundary-Layer Turbulence With Mass Addition and Combustion," *AIAA Journal,* Vol. 4, Nov. 1966, pp. 2009–2016.

[6] Klebanoff, D. S., "Characteristics of Turbulence in a Boundary Layer with Zero Pressure Gradient," TN 3178, July 1954, NACA.

[7] Simpson, R. L., "The Turbulent Boundary Layer on a Porous Plate: An Experimental Study of the Fluid Mechanics With Injection and Suction," Ph.D. thesis, 1967, Dept. of Mechanical Engineering, Stanford University, Palo Alto, Calif.

Effects of Constraint Modification on the Random Vibration of Damped Linear Structures

L. J. Howell*

General Motors Corporation, Warren, Mich.

Introduction

MOTIVATED by a desire for enhanced computing efficiency, numerous investigators have recently explored techniques for assessing the effects of system modification on structural vibration. Such techniques could also provide a foundation upon which efficient optimization algorithms might be developed. Much of the reported research has dealt with procedures necessary to determine the eigenvalues and eigenvectors of the modified system by utilizing available information concerning the original system. For example, such methods which are applicable for treating damped linear systems are discussed in Ref. 1.

Dowell[2] has described a procedure for computing the vibration modes of a structure whose support conditions are modified. The method uses the normal modes of the original system in a Rayleigh-Ritz analysis with the new constraints enforced by Lagrange multipliers. His technique has obvious potential for treating the synthesis of large structures whose components are defined in terms of their unconstrained modes.[3] Hallquist and Snyder[4] have recently extended Dowell's work to include viscous damping.

In the present Note, we have utilized the results of Refs. 2 and 4 to formulate a direct solution for the stationary random response of the modified system. Specifically, the matrix of cross-spectral density functions for the modified system is written in terms of the cross-spectral density matrix of the original system, the frequency response functions for the original system, and the parameters describing the added constraints. These new results are believed to have significant practical application, particularly since computer programs could be used to generate directly the transfer functions and cross-spectral density functions—these might then be stored and used in a post-processor scheme to determine the random vibration of the modified system.

Received July 3, 1974; revision received November 11, 1974. The author wishes to express his appreciation to the reviewers for the comments and helpful suggestions which were used in preparing this Note.

Index category: Structural Dynamic Analysis.

* Supervisory Research Engineer, Engineering Mechanics Department. Member AIAA.

Analysis

Consider a dynamic system with n degrees of freedom. Let the characteristics of the linear system be defined by the generalized mass, stiffness, and damping matrices $[M]$, $[K]$, and $[C]$. Assume that the structure is to be constrained by g rigid supports and h elastic supports. The displacements of the nonrigid supports at the support-structure attachment points will be designated $\{p\}$. The support modifications can be described by the k constraint relations:

$$\{f\} = [\beta]\{q\} - \begin{Bmatrix} \{0\} \\ \{p\} \end{Bmatrix} = \{0\} \tag{1}$$

where $[\beta]$ defines the type of modification and $\{q\}$ is the vector of generalized coordinates for the unconstrained (or less constrained) structure. Note that $k = g+h$. A Lagrangian formulation yields n equations of the form[2,4]

$$[M]\{\ddot{q}\}+[C]\{\dot{q}\}+[K]\{q\}-[\beta]^T\{\lambda\} = \{Q\} \tag{2}$$

and h equations

$$\lceil K_s \rfloor \{p\} = -\{\lambda_1\} \tag{3}$$

where Q_i are the generalized forces and $\lceil K_s \rfloor$ defines the stiffnesses of the h elastic constraints. (For simplicity we have assumed that the elastic constraints are individually grounded.) The total set of Lagrange multipliers, $\{\lambda\}$, consists of those which define rigid constraint forces $\{\lambda_0\}$ and those which define elastic constraint forces $\{\lambda_1\}$. Thus,

$$\{\lambda\} = \begin{Bmatrix} \{\lambda_0\} \\ \{\lambda_1\} \end{Bmatrix}$$

The applicable constraint relations, k in number, are

$$[\beta]\{q\} = \begin{Bmatrix} \{0\} \\ \{p\} \end{Bmatrix} \tag{4}$$

where $\{p\}$ may be written in terms of the Lagrange multipliers $\{\lambda_1\}$ by using Eq. (3).

Usually, Eq. (2) will be coupled because of the presence of off-diagonal terms in the mass, stiffness, or damping matrices. However, the system can be uncoupled by introducing the transformation[5]

$$\{z\} = \begin{Bmatrix} \{\dot{q}\} \\ \{q\} \end{Bmatrix} = [\Phi]\{\xi\} \tag{5}$$

Equations (2) and (4) then become[4]

$$\lceil I \rfloor \{\dot{\xi}\} - \lceil \mu \rfloor \{\xi\} - [W]^T\{\lambda\} = \{\bar{F}\} \tag{6}$$

$$[W]\{\xi\} = \begin{Bmatrix} \{0\} \\ \{p\} \end{Bmatrix} \tag{7}$$

where $[\ulcorner I \lrcorner]$ is the identity matrix, T denotes transposition,

$$[W] = ([0][\beta])[\Phi],\ \{F\} = [\Phi]^T \begin{Bmatrix} \{0\} \\ \{Q\} \end{Bmatrix},\ [\Phi]^T[A][\Phi] = [\ulcorner I \lrcorner]$$

and

$$[\Phi]^T[B][\Phi] = -[\ulcorner \mu \lrcorner]$$

Notice that $[\Phi]$ and $[\ulcorner \mu \lrcorner]$ are the matrices of eigenvectors and eigenvalues, respectively, for the homogeneous equations

$$[A]\{\dot{Z}\} + [B]\{Z\} = \{0\}$$

where

$$[A] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix} \qquad [B] = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix}$$

Also, the columns of $[\Phi]$ may be written[5]

$$\{\Phi\} = \begin{Bmatrix} \{\theta\}\mu \\ \{\theta\} \end{Bmatrix}$$

For the case of free vibration of the modified system, that is when $\{F\} = \{0\}$ and harmonic motion is assumed, Eqs. (6) and (7) lead to the result previously given in Ref. 4. That result is the characteristic equation for the $2n$ complex eigenvalues ω_j of the constrained system. That characteristic equation is obtained by forcing the determinant of coefficients of the Lagrange multipliers to vanish. When $\{p\} = \{0\}$, the result is:

$$\left| [W] \left[\diagdown \frac{1}{\omega - \mu_j} \diagdown \right] [W]^T \right| = 0 \tag{8}$$

In the present Note, we are interested in extending previous results so that the statistics of the structural response of the modified system can be written directly in terms of the properties and response of the original system and the constraint parameters $[\beta]$ and $[\ulcorner K_s \lrcorner]$. In particular, the matrix of power spectral density functions for the modified system will be derived.

To begin the development, compute the transfer functions for the modified system. Let

$$Q = \bar{Q} e^{i\omega t}$$

in Eq. (6); then $\{\lambda\}$, $\{\xi\}$, and $\{p\}$ become

$$\{\lambda\} = \{\bar{\lambda}\}\, e^{i\omega t}, \quad \{\xi\} = \{\bar{\xi}\}\, e^{i\omega t}, \quad \{p\} = \{\bar{p}\}\, e^{i\omega t}$$

Solving Eq. (6) for $\{\bar{\xi}\}$ gives

$$\{\bar{\xi}\} = \left[\diagdown \frac{1}{i\omega - \mu_j} \diagdown \right] \left[[W]^T\{\bar{\lambda}\} + [\Phi]^T \begin{Bmatrix} \{0\} \\ \{\bar{Q}\} \end{Bmatrix} \right] \tag{9}$$

Substituting Eq. (9) into the constraint equations yields the Lagrange multipliers:

$$\{\bar{\lambda}\} = -[R]^{-1}[W] \left[\diagdown \frac{1}{i\omega - \mu_j} \diagdown \right] [\Phi]^T \begin{Bmatrix} \{0\} \\ \{\bar{Q}\} \end{Bmatrix} \tag{10}$$

where

$$[R] = [W] \left[\diagdown \frac{1}{i\omega - \mu_j} \diagdown \right] [W]^T + [\ulcorner T \lrcorner]$$

As a consequence of the assumptions concerning the nonrigid supports $[\ulcorner T \lrcorner]$ is a diagonal matrix which may be partitioned as follows

$$[\ulcorner T \lrcorner] = \begin{bmatrix} [0] & [0] \\ [0] & [\ulcorner K_s^{-1} \lrcorner] \end{bmatrix}$$

Substituting Eq. (10) into Eq. (9) gives

$$\{\bar{\xi}\} = \left[\diagdown \frac{1}{i\omega - \mu_j} \diagdown \right] \left[[I] - [W]^T[R]^{-1}[W] \left[\diagdown \frac{1}{i\omega - \mu_j} \diagdown \right] \right] \times [\Phi]^T \begin{Bmatrix} \{0\} \\ \{\bar{Q}\} \end{Bmatrix} \tag{11}$$

Equation (11) can be put into a more useful form by transforming back to the generalized coordinates q. First, premultiply Eq. (11) by $[\Phi]$. Then replace $[W]$ by $[[0][\beta]][\Phi]$ and $[\Phi]$ by

$$\begin{bmatrix} [\theta][\ulcorner \mu \lrcorner] \\ [\theta] \end{bmatrix}$$

After expanding Eq. (11) and comparing it with Eq. (5), it can be seen that

$$\{\bar{q}\} = [[\ulcorner I \lrcorner] - [H_o(\omega)][\beta]^T[R]^{-1}[\beta]]\,[H_o(\omega)]\{\bar{Q}\} \tag{12}$$

where

$$[H_o(\omega)] = [\theta] \left[\diagdown \frac{1}{i\omega - \mu} \diagdown \right] [\theta]^T$$

Notice that $[H_o(\omega)]$ is equal to $\{\bar{q}\}/\{\bar{Q}\}$ prior to constraint modification. $H_o(\omega)$ is, therefore, the transfer function for the original system. The coefficient of $\{\bar{Q}\}$ in Eq. (12) is the matrix of transfer functions for the constrained system.

In many structural applications the original system is underdamped; hence, oscillatory. In this case, the complex eigenvalues occur in conjugate pairs so that $[\ulcorner \mu \lrcorner]$ and $[\Phi]$ can be written[6,7]

$$[\ulcorner \mu \lrcorner] = \begin{bmatrix} [\ulcorner \gamma \lrcorner] & [0] \\ [0] & [\ulcorner \gamma^* \lrcorner] \end{bmatrix} \qquad [\Phi] = \begin{bmatrix} [\phi][\ulcorner \gamma \lrcorner] & [\phi^*][\ulcorner \gamma^* \lrcorner] \\ [\phi] & [\phi^*] \end{bmatrix}$$

where $[\theta] = [\phi \,\vdots\, \phi^*]$ and the asterisk denotes conjugation. In this special case $[H_o(\omega)]$ becomes

$$[H_o(\omega)] = [\phi] \left[\diagdown \frac{1}{i\omega - \gamma_j} \diagdown \right] [\phi]^T + [\phi^*] \left[\diagdown \frac{1}{i\omega - \gamma_j^*} \diagdown \right] [\phi^*]^T$$

Returning now to the general formulation, we will compute the effects of constraint modification on the vibration response when the system is exposed to random exciting forces. The matrix of cross-spectral density functions can be written:

$$[S_{q_i q_j}(\omega)] = [H^*(\omega)]\,[S_{P_i P_j}(\omega)]\,[H(\omega)]^T \tag{13}$$

where $[S_{P_i P_j}]$ is the cross-spectral density matrix describing the applied stationary random loading and $[H(\omega)]$ is the transfer function. The matrix element $S_{q_i q_j}$ denotes the cross-spectral density function of the output quantities q_i and q_j. If we let $[H(\omega)]$ in Eq. (13) represent the transfer function for the modified system, given as the coefficient of $\{\bar{Q}\}$ in Eq. (12), we see that

$$[S_{q_i q_j}(\omega)] = [[I] - [H_o^*(\omega)][\beta^*]^T[R^*]^{-1}[\beta^*]]\,[H_o^*(\omega)] \cdot [S_{P_i P_j}(\omega)]\,[H_o(\omega)]^T[[\ulcorner I \lrcorner] - [H_o(\omega)][\beta]^T[R]^{-1}[\beta]]^T \tag{14}$$

If we define the matrix of cross-spectral density functions for the response of the unconstrained structure as

$$[S^o_{q_i q_j}(\omega)] = [H_o^*(\omega)]\,[S_{P_i P_j}(\omega)]\,[H_o(\omega)]^T$$

then it can be shown by expanding Eq. (14) that the cross-spectral density matrix for the new system is

$$[S_{q_i q_j}(\omega)] = [S^o_{q_i q_j}(\omega)] - [D^*(\omega)]\,[S^o_{q_i q_j}(\omega)] - [S^o_{q_i q_j}(\omega)]\,[D(\omega)]^T + [D^*(\omega)]\,[S^o_{q_i q_j}(\omega)]\,[D(\omega)]^T \tag{15}$$

where

$$[D(\omega)] = [H_o(\omega)][\beta]^T[R]^{-1}[\beta]$$

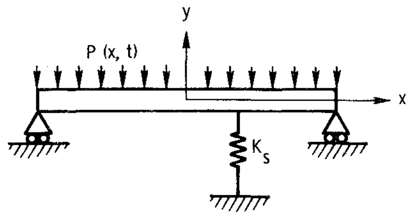

Fig. 1 Simply supported beam with an added elastic constraint.

Illustrative Example

In many practical situations, the coordinates used to describe the original structure will be its normal coordinates. Furthermore, it is frequently of interest to investigate the modified response when only a single constraint is added. Then, only a single Lagrange multiplier is needed, $[\beta]$ reduces to a row matrix, and $[R]$ reduces to a scalar thereby eliminating the need for matrix inversion in either Eq. (12) or (15). As an example problem of this type, consider the simply supported beam shown in Fig. 1. The cross-spectral density for the generalized forces in modes j and k is[8]

$$F_{jk} = \int\int S_{PP}(x_1, x_2; \omega)\, \phi_j(x_1)\, \phi_k(x_2)\, dx_1\, dx_2$$

where $\phi_j(x)$ are the orthogonal modes for the original structure. For a spatially uncorrelated, weakly stationary pressure loading, the matrix for cross-spectral densities for the generalized forces becomes

$$\lceil F_{jj}(\omega) \rfloor$$

where

$$F_{jj}(\omega) = \left[\int \phi_j(x)\, dx\right]^2 S_{PP}(\omega)$$

The cross-spectral density matrix describing the modal response of the simply supported beam is

$$\lceil S^o_{q_j q_j}(\omega) \rfloor = \lceil H_o^*(\omega) \rfloor \lceil F_{jj}(\omega) \rfloor \lceil H_o(\omega) \rfloor$$

where $\lceil H_o(\omega) \rfloor$ is the matrix of frequency response functions describing the modal response of the original structure.

Now consider the addition of the elastic support with stiffness equal to K_s. The support is located at $x = x_o$. Thus the constraint relation is

$$f_1 = w(x_o, t) - p_1 = 0$$

where $w(x, t)$ is the beam vertical deflection and p_1 is the displacement of the elastic support at its beam attachment point. From the expression

$$w(x, t) = \sum_{i=1}^{n} \phi_i(x)\, q_i(t)$$

we see that

$$f_1 = \sum_{i=1}^{n} \phi_i(x_o)\, q_i(t) - p_1 = 0$$

Therefore the constraint parameters in Eq. (15) are

$$\beta_j = \phi_j(x_o) \qquad j = 1, 2, \ldots, n$$

Also, $\{\lambda\} = \lambda_1$ since there are no added rigid constraints and only one elastic constraint. Therefore, Eq. (3) becomes

$$K_s p_1 = -\lambda_1$$

and $\lceil T \rfloor$ reduces to $T = K_s^{-1}$. $[R]$ reduces to the scalar function of frequency

$$R(\omega) = \sum_{j=1}^{n} \phi_j^2(x_o)\, H_j(\omega) + K_s^{-1}$$

Evaluating the cross-spectral density matrix for modal response yields, after extensive algebraic manipulation

$$S_{q_s q_s}(\omega) = \left\{ S^o_{q_s q_s}(\omega) \left| \sum_{j \neq s}^{n} \phi_j^2 H_j^* + K_s^{-1} \right|^2 + |H_s|^2 \phi_s^2 \sum_{l \neq s}^{n} \phi_l^2 S^o_{q_l q_l}(\omega) \right\} \Big/ |R(\omega)|^2$$

for $t = s$, and

$$S_{q_s q_t}(\omega) = \phi_s \phi_t \left\{ H_s^* H_t \sum_{j \neq s,t}^{n} \phi_j^2 S^o_{q_j q_j}(\omega) - S^o_{q_s q_s}(\omega) H_t \times \left(K_s^{-1} + \sum_{j \neq s}^{n} \phi_j^2 H_j^* \right) - S^o_{q_t q_t}(\omega) H_s^* \left(K_s^{-1} + \sum_{j \neq t}^{n} \phi_j^2 H_j \right) \right\} \Big/ |R(\omega)|^2$$

for $t \neq s$ and for notational convenience, the terms of the diagonal matrix $\lceil H_o(\omega) \rfloor$ have been written H_j, and $\phi_i(x_o)$ has been written ϕ_i. Notice that the right-hand sides of these equations contain only those terms which characterize either the constraint parameters, or the system and its response prior to constraint addition. Those terms which characterize the original system and its response could easily be stored during an initial computer analysis and utilized later to investigate constraint modification.

Conclusions

An expression has been given for computing the matrix of cross-spectral density functions for a system which is subjected to constraint modification. Second-order response statistics for the displacement of the modified structure could be determined in the usual manner once the appropriate cross-spectral density matrix has been generated. Although not shown explicitly in this Note, it appears that the technique can be generalized to investigate system modification.

References

[1] Pomazal, R. J. and Snyder, V. W., "Local Modification of Damped Linear Systems," *AIAA Journal*, Vol. 9, Nov. 1971, pp. 2216–2221.

[2] Dowell, E. H., "Free Vibrations of a Linear Structure with Arbitrary Support Conditions," *Journal of Applied Mechanics*, Vol. 38, Ser. E, Sept. 1971, pp. 595–600.

[3] Dowell, E. H., "Free Vibrations of an Arbitrary Structure in terms of Component Modes," *Journal of Applied Mechanics*, Vol. 39, Ser. E, Sept. 1972, pp. 727–732.

[4] Hallquist, J. and Snyder, V. W., "Linear Damped Vibratory Structures with Arbitrary Support Conditions," *Journal of Applied Mechanics*, Vol. 40, Ser. E, March 1973, pp. 312–313.

[5] Foss, K. A., "Coordinates Which Uncouple the Equations of Motion of Damped Linear Dynamic Systems," *Journal of Applied Mechanics*, Vol. 25, Sept. 1958, pp. 361–364.

[6] Hurty, W. C. and Rubinstein, M. F., *Dynamics of Structures*, Prentice-Hall, Englewood Cliffs, N.J., 1964, p. 320.

[7] Hasselman, T. K., "Method of Constructing a Full Modal Damping Matrix from Experimental Measurements," *AIAA Journal*, Vol. 10, April 1972, pp. 526–527.

[8] Lin, Y. K., *Probabilistic Theory of Structural Dynamics*, McGraw-Hill, New York, N.Y., 1967, p. 211.

Flutter of a Panel Supported on an Elastic Foundation

INDERJIT CHOPRA*
National Aeronautical Laboratory, Bangalore, India

Introduction

A PANEL supported on an elastic medium often finds application in the construction of aerospace structures. This elastic medium can be any springy material such as a heat shield or damping tape attached to one side of the panel; the other side of the panel is exposed to the air flow. Dugundji, Dowell, and Perkin[1] have analyzed the flutter of a long panel resting on a linear elastic foundation in subsonic flow using the theory due to Miles.[2] They found that flutter is possible for a particular panel configuration in subsonic flow, and this flutter is mainly of a traveling-wave type. These results were also confirmed by experiment. Johns[3] studied the static aeroelastic instability of orthotropic rectangular panels resting on an elastic medium for low speed as well as supersonic flows. He has shown that the elastic medium affects significantly the aeroelastic divergence of a panel in incompressible flow. McElman,[4] as well as Johns and Taylor[5] investigated the flutter behavior of two flat panels connected by a linear elastic medium, where the flow is exposed to only one surface of the construction. They found that the response of the elastic medium, in conjunction with inplane forces in the panels, strongly affects the flutter characteristics. Dugundji,[6] who has made an exhaustive analysis of the

Received July 3, 1974; revision received November 8, 1974. The author wishes to acknowledge useful discussion with Professor J. Dugundji and M. Pustejovsky of the Massachusetts Institute of Technology.

Index category: Panel Flutter.

* Scientist, Structural Sciences Division. Presently Graduate Student in the Department of Aeronautics and Astronautics, Massachusetts Institute of Technology.